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# Solution of Two-Dimensional, Two-Region Electromagnetic Ground Response

by Ira Kohlberg



Prepared by

Kohlberg Associates, Inc.
P.O. Box 23077
Alexandria, VA 22304

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A solution for the two-dimensional, two-region electromagnetic ground response has been developed which relates the surface components of the electric field to the surface components of the magnetic field. This has been accomplished by deriving a universal functional form for a dimensionless Green's function. The Green's function provides increasingly more accurate approximations to the response for each successive reflection from the second layer. This result would appear to provide simplification and reduced computer running time in the numerical modelling of the HABEMP when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem.

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## 1. INTRODUCTION

In a previous study[1] we showed that for an $n$-layer earth model it is possible to express the three components of the electric field and the vertical component of the magnetic field on the surface of the earth as a space-time integration of the two horizontal components of the magnetic field. In particular, it was shown that if $\vec{r_s}$ is a point on the surface ($x, y$ plane), and if $Y_i$ is a member of the set

$$Y_i(\vec{r_s}, t) = \{H_z, E_x, E_y, E_z\} \quad , \tag{1.1}$$

then every member of the set $Y_i$ on the surface of a finitely conducting earth can be related to the horizontal components of the surface magnetic field through the equation

$$\begin{aligned} Y_1(\vec{r_s}, t) = & \int_0^t \int_{\vec{r_s'}} G_{ix}(\vec{r_s} - \vec{r_s'}, t - t') H_x(\vec{r_s'}, t')\ dx'\ dy'\ dt' \\ & + \int_0^t \int_{\vec{r_s'}} G_{iy}(\vec{r_s} - \vec{r_s'}, t - t') H_y(\vec{r_s'}, t')\ dx'\ dy'\ dt' \quad . \end{aligned} \tag{1.2}$$

The functions $G_{ix}$ and $G_{iy}$ are Green's functions, which are determined from the solution of the $n$-region ground model shown in figure 1.

The result given by equation (1.2) can possibly provide considerable simplification in the numerical modelling of the high-altitude burst electromagnetic pulse (HABEMP) when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem. When this approach can be used it obviates the necessity of developing a numerical representation of the ground, which then reduces the number of variables in the problem and hence the computer running time (and cost). On the other hand, the reduction of machine variables must be weighed against the speed of the numerical computation for the integral boundary conditions arising from the Green's function formalism. This ques-

(1) I. Kohlberg, Surface Integral Representation of Three Dimensional Electromagnetic Ground Response for Multi-Layered Earth With Frequency Dependent Electrical Parameters, April 1986, Final Report for Harry Diamond Laboratories, Contract No. DAAG29-81-D-010, Delivery Order 2064, Battelle Columbus Laboratories (Prime Contractor).

tion is as yet unresolved; in any event it will depend on the number of ground layers being considered, the values of the electrical parameters, and the time range of interest.

Equation (1.2) is derived by solving the $n$-region three-dimensional problem in the Fourier (space) and Laplace (time) domains and ultimately performing inverse Fourier/Laplace transforms. (The reader is referred to reference 1 for details.) Although the formalism and mathematical procedure are general, the analytical Fourier/Laplace transform inversion may not always be possible. For the purposes of assessing the feasibility of using the theory of reference 1, four special-case solutions were developed. These included the following models:

(a) one layer, one dimension,

(b) one layer, two dimensions,

(c) one layer, three dimensions, and

(d) two layers, one dimension.

In this study we extend the number of solutions to include the two-layer two-dimensional case. The geometry for this case is shown in figure 2. Following an earlier analysis,[1] we consider the case where the displacement current in the ground is less than the conduction current. We assume that spatial variations in the $y$-direction are neglected. Our principal concern is to establish the space-time relationships between $E_x, E_y$ and the driving functions, $H_x, H_y$.

## 2. MATHEMATICAL NOTATION

As will become evident in sections 3 and 4 of this report, several mathematical operations are performed on the four surface values of the fields used in this analysis: namely, $E_x(x,t)$, $E_y(x,t)$, $H_x(x,t)$, and $H_y(x,t)$. The purpose of this section is to define these operations and develop a shorthand notation for dealing with them. If $F(x,t)$ represents any one of the aforementioned four functions, the Fourier and Laplace transforms are defined by the following oprrations:

$$\hat{F}(k,t) = \underline{M}F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}F(x,t)\,dx \quad , \tag{2.1}$$

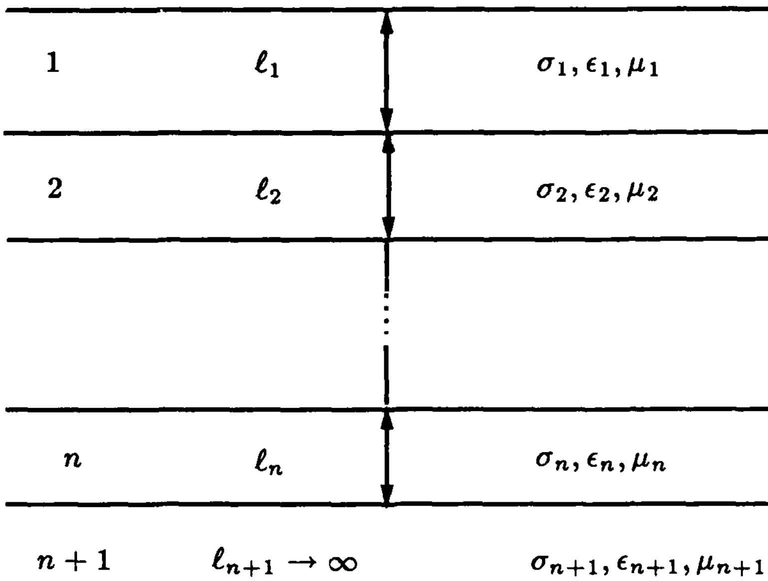
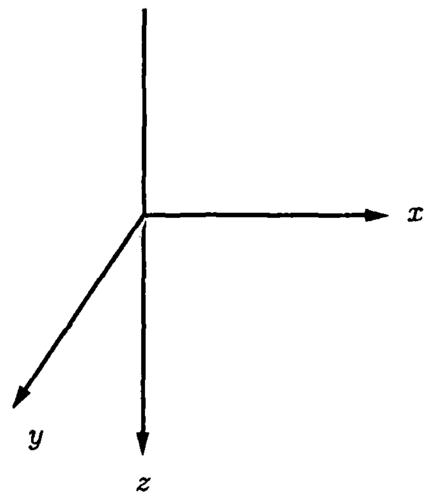



Figure 1. $(n+1)$-region model.



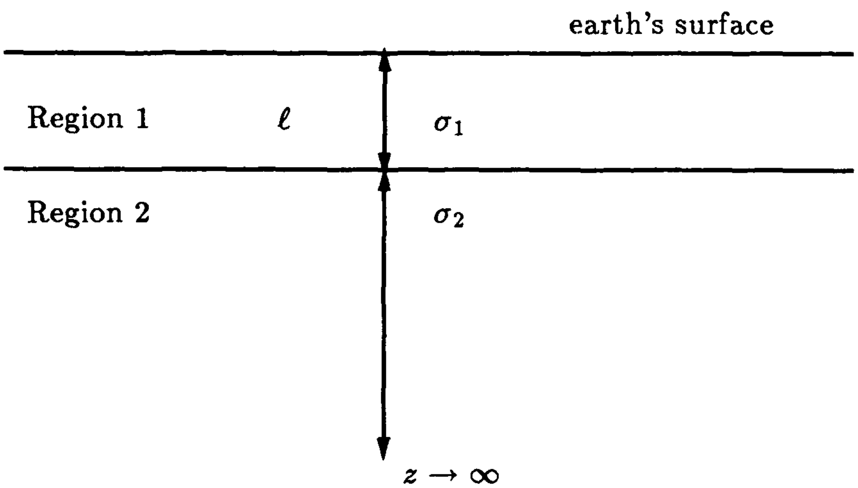




Figure 2. Geometric considerations.

$$\tilde{F}(x,s) = \underline{L}F(x,t) = \int_0^\infty e^{-st} F(x,t)\ dt \quad , \tag{2.2}$$

$\underline{M}$ = Fourier transform operator, and

$\underline{L}$ = Laplace transform operator.

The double-transformed function, $\bar{F}(k,s)$, is given by

$$\begin{aligned} \bar{F}(k,s) &= \underline{M}\,\underline{L}F(x,t) = \underline{L}\,\underline{M}F(x,t) \\ &= \frac{1}{2\pi} \int_0^\infty e^{-st} \int_{-\infty}^\infty e^{-ikx} F(x,t)\ dx\ dt \quad . \end{aligned} \tag{2.3}$$

As indicated in equation (2.3), the order in which the transforms are taken is immaterial.

The inverse Fourier and Laplace operators are defined by operations

$$\underline{M}^{-1}\hat{F}(k,t) = \int_{-\infty}^\infty e^{ikx} \hat{F}(k,t)\ dk = F(x,t) \quad , \tag{2.4}$$

$$\underline{L}^{-1}\tilde{F}(x,s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{F}(x,s)\ ds = F(x,t) \quad . \tag{2.5}$$

It also follows that

$$\underline{M}^{-1}\bar{F}(k,s) = \tilde{F}(x,s) \quad , \tag{2.6}$$

$$\underline{L}^{-1}\bar{F}(k,s) = \hat{F}(k,t) \quad . \tag{2.7}$$

Using the Faltung theorem we can write, for any functions $\bar{f}_1(k,s)$ and $\bar{f}_2(k,s)$,

$$\begin{aligned} \underline{M}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) &= \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}_1(x-x',s)\tilde{f}_2(x',s)\ dx' \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}_2(x-x',s)\tilde{f}_1(x',s)\ dx' \quad . \end{aligned} \tag{2.8}$$

The shorthand notation for the operation of equation (2.8) is

$$\underline{M}^{-1}\left(\bar{\tilde{f}}_1(k,s)\bar{\tilde{f}}_2(k,s)\right) = \tilde{f}_1(x,s) \otimes \tilde{f}_2(x,s) \quad , \tag{2.9}$$

where the "$\otimes$" stands for the space integration of equation (2.8). It also readily follows that

$$\underline{M}^{-1}\left(\hat{f}_1(k,t)\hat{f}_2(k,t)\right) = f_1(x,t) \otimes f_2(x,t) \quad . \tag{2.10}$$

The convolution theorem for the Laplace transform yields the following result:

$$\begin{aligned} \underline{L}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) &= \int_0^t \hat{f}_1(k,t-t')\hat{f}_2(k,t')\, dt' \\ &= \int_0^t \hat{f}_2(k,t-t')\hat{f}_1(k,t')\, dt' \quad . \end{aligned} \tag{2.11}$$

The shorthand notation for the convolution integration in equation (2.11) is

$$\underline{L}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) = \hat{f}_1(k,t) * \hat{f}_2(k,t) \quad , \tag{2.12}$$

where "$*$" denotes the convolution operation. We also have

$$\underline{L}^{-1}\left(\tilde{f}_1(x,s)\tilde{f}_2(x,s)\right) = f_1(x,t) * f_2(x,t) \quad . \tag{2.13}$$

The mathematical formalism and notation developed through equation (2.13) provides a compact way of identifying the Green's function which relates the surface values of the electric field to the magnetic field components. For example, as shown in section 3, we have the relationship

$$\bar{E}_y(k,s) = \bar{G}(k,s)\bar{H}_x(k,s) \quad . \tag{2.14}$$

Using equations (2.8) and (2.9) we can write

$$\underline{M}^{-1}\,\underline{L}^{-1}\bar{E}_y(k,s) = \underline{M}^{-1}\hat{E}_y(k,t) = E_y(x,t) \quad . \tag{2.15}$$

If we identify $\bar{G}(k,s)$ as $\bar{f}_1(k,s)$ and $\bar{H}_x(k,s)$ as $\bar{f}_2(k,s)$, we deduce the relationship

$$\begin{aligned}
\underline{M}^{-1}\,\underline{L}^{-1}\bar{E}_y(k,s) &= \underline{M}^{-1}\,\underline{L}^{-1}\left(\bar{G}(k,s)\bar{H}_x(k,s)\right) \\
\bar{E}_y(x,t) &= G(x,t)\otimes * H_x(x,t) \\
\bar{E}_y(x,t) &= \int_{-\infty}^{\infty}\int_0^t G(x-x',t-t')H_x(x',t')\;dx'\;dt' \qquad (2.16)
\end{aligned}$$

where

$$\begin{aligned}
G(x,t) &= \underline{M}^{-1}\,\underline{L}^{-1}\bar{G}(k,s) = \underline{L}^{-1}\,\underline{M}^{-1}\bar{G}(k,s) \\
&= \text{Green's function} \quad . \qquad (2.17)
\end{aligned}$$

Employing the techniques of reference 1 we can derive an expression for $\bar{G}(k,s)$. The thrust of this investigation is to develop techniques for determining $G(x,t)$ from the inverse Fourier/Laplace transforms.

## 3. EQUATIONS AT SURFACE OF EARTH

It is shown in equation (3.21) of reference 1 that for the $n$th region of an $n$-region ground model, the equations for $\bar{E}_{x,n}(z,k_x,k_y,s)$ and $\bar{E}_{y,n}(z,k_x,k_y,s)$ in terms of the magnetic field components, $\bar{H}_{x,n}(z,k_x,k_y,s)$, $\bar{H}_{y,n}(z,k_x,k_y,s)$, are given by

$$\bar{E}_{x,n} = \frac{\gamma_n}{\sigma_n}\left[-\bar{H}_{y,n} + \frac{1}{\gamma_n^2}(k_y k_x \bar{H}_{x,n} + k_y^2 \bar{H}_{y,n})\right] \quad , \qquad (3.1)$$

$$\bar{E}_{y,n} = \frac{\gamma_n}{\sigma_n}\left[\bar{H}_{x,n} - \frac{1}{\gamma_n^2}(k_x^2 \hat{H}_{x,n} + k_x k_y \hat{H}_{y,n})\right] \quad , \qquad (3.2)$$

where $\sigma_n$ = conductivity and

$$\gamma_n = \pm\lambda_n \qquad (3.3)$$

$$\lambda_n = \sqrt{s\mu_n\sigma_n + k_x^2 + k_y^2} \quad , \qquad (3.4)$$

with + being used for the upward wave, and − being used for the downward wave.

We immediately note from equations (3.1) and (3.2) that if either $k_y$ or $k_x$ is equal to zero (this corresponds to neglecting spatial variations in the $y$ and $x$ directions, respectively) the equations are decoupled into two independent pairs. For example, setting $k_y = 0$ gives

$$\bar{E}_{x,n} = -\frac{\gamma_n}{\sigma_n}\bar{H}_{y,n} \quad , \tag{3.5}$$

$$\bar{E}_{y,n} = \frac{\gamma_n}{\sigma_n}\left(1 - \frac{k_x^2}{\gamma_n^2}\right)\bar{H}_{x,n} \quad , \tag{3.6}$$

with $\gamma_n$ now being given by

$$\gamma_n = \pm\sqrt{s\mu_n\sigma_n + k_x^2} \quad . \tag{3.7}$$

When equations (3.5) to (3.7) are used in the solution of the two-region problem of figure 2, it can be shown that the surface field equations are given by

$$\bar{E}_x = r_1\left[\bar{H}_y + 2\sum_{n=1}^{\infty}\bar{\Psi}_r^n\bar{H}_y\right] \quad , \tag{3.8}$$

$$\bar{E}_y = -w_1\left[\bar{H}_x + 2\sum_{n=1}^{\infty}\bar{\Psi}_w^n\bar{H}_x\right] , \tag{3.9}$$

where

$$\begin{aligned}
r_1 &= \frac{\lambda_1}{\sigma_1} \quad , \qquad r_2 = \frac{\lambda_2}{\sigma_2} \quad , \\
w_1 &= \frac{s\mu}{\lambda_1} \quad , \qquad w_2 = \frac{s\mu}{\lambda_2} \quad , \\
\lambda_1 &= \sqrt{s\mu\sigma_1 + k^2} \quad , \qquad \lambda_2 = \sqrt{s\mu\sigma_2 + k^2} \quad , \\
\bar{\Psi}_r &= \frac{r_2 - r_1}{r_2 + r_1}\exp(-2\lambda_1 l) \quad , \\
\bar{\Psi}_w &= \frac{w_2 - w_1}{w_2 + w_1}\exp(-2\lambda_1 l) \quad .
\end{aligned} \tag{3.10}$$

For brevity we have replaced $k_x$ by $k$, it being understood that we only considering spatial variations in the $x$ direction.

Equations (3.8) and (3.9) reduce to the one-layer, two-dimensional case considered in reference 1 when $\sigma_2 = \sigma_1$. In this situation, $\bar{\Psi}_r$ and $\bar{\Psi}_w$ are both equal to zero, and we obtain the solutions

$$\bar{E}_{x,0} = r_1 \bar{H}_y \quad , \tag{3.11}$$

$$\bar{E}_{y,0} = -w_1 \bar{H}_x \quad . \tag{3.12}$$

The space-time behavior of $E_{y,0}(x,t)$ is given by[1]

$$\begin{aligned} E_{y,0} &= \underline{M}^{-1}\,\underline{L}^{-1}\bar{E}_{y,0} \\ &= -\frac{\mu}{2\pi}\int_0^t \int_{-\infty}^{\infty} \frac{1}{t-t'} \exp\left[-\frac{(x-x')^2 \mu\sigma_1}{4(t-t')}\right] \left(\frac{\partial}{\partial t'} H_x(x',t')\right) dx'\, dt' \quad . \end{aligned} \tag{3.13}$$

For the one-layer, two-dimensional case, equation (3.13) represents a means of establishing the boundary condition on top of the earth. This equation can be written in finite-difference form and thus can be used with the HABEMP equations above the earth's surface to provide a self-consistent representation of the overall physical model. The key question is "Under what conditions is the method more efficient than representing the ground in a finite-difference approximation?" The answer to this will be forthcoming in the near future.

For the two-layer, two-dimensional case being considered here, the deduction of $E_y(x,t)$ in terms of $H_x(x,t)$ is more complicated than that of equation (3.13) because of the complexity of $\bar{\Psi}_w(k,s)$. A similar statement can be made for the $E_x, H_y$ pair; however, for brevity this pair is not being considered since the analysis is similar to the $E_y, H_x$ case.

It is possible to express the solution of equation (3.9) in several different ways, leading in turn to different algorithms for completing the calculation of $E_y(x,t)$. For example, performing the $\underline{L}^{-1}\,\underline{M}^{-1}$ operation directly on equation (3.9) leads to the result

$$E_y = E_{y,0} + 2\sum_{n=1}^{\infty} E_{y,n} \quad , \tag{3.14}$$

where

$$E_{y,n} = G_n(x,t) \otimes *H_x(x,t) \ , \quad n \geq 1 \ , \tag{3.15}$$

$$G_n(x,t) = \underline{L}^{-1}\,\underline{M}^{-1}\left(-\bar{\Psi}_w^n w_1\right) \quad . \tag{3.16}$$

An alternative and mathematically equivalent formalism is based on using the knowledge of $E_{y,0}$ deduced from equation (3.13). We can write equation (3.9) in the form

$$\bar{E}_y = \bar{E}_{y,0} + 2\sum_{n=1}^{\infty} \bar{\Psi}_w^n \bar{E}_{y,0} \quad , \tag{3.17}$$

which then leads to the relationship

$$E_y = E_{y,0} + 2\sum_{n=1}^{\infty} E_{y,n} \quad . \tag{3.18}$$

$E_{y,n}$ is expressed as

$$E_{y,n} = G_n(x,t) \otimes *E_{y,0}(x,t) \quad , \tag{3.19}$$

with $G_n(x,t)$ now given by

$$G_n(x,t) = \underline{L}^{-1}\,\underline{M}^{-1}\left(\bar{\Psi}_w^n\right) \quad . \tag{3.20}$$

The analysis of this investigation is concerned with the determination of $G_n(x,t)$ from equation (3.19).

There is, however, one additional method that can be used, which is related to equation (3.20). We include this for completeness. Consider, for example, the sequence of functions

$$\bar{E}_{y,1} = \bar{\Psi}_w \bar{E}_{y,0}$$

$$\bar{E}_{y,2} = \bar{\Psi}_w^2 \bar{E}_{y,0} = \bar{\Psi}_w \bar{E}_{y,0}$$

$$.$$

$$.$$

$$.$$

$$\bar{E}_{y,n} = \bar{\Psi}_w \bar{E}_{y,n-1} \tag{3.21}$$

Using equation (3.21) we can write

$$E_y = E_{y,0} + 2 \sum_{n=1}^{\infty} E_{y,n} \quad . \tag{3.22}$$

$E_{y,n}$ is now given by

$$E_{y,n} = G_1 \otimes * E_{y,n-1} \tag{3.23}$$

with

$$G_1(x,t) = \Psi_w(x,t) = \underline{L}^{-1}\, \underline{M}^{-1} \bar{\Psi}_w(k,s) \quad . \tag{3.24}$$

It is also observed by comparing equations (3.20) and (3.24) that $G_n(x,t)$ is the $(n-1)^{st}$ space-time convolution of $\Psi_w(x,t)$. That is,

$$G_2 = \underline{L}^{-1}\, \underline{M}^{-1} \left(\bar{\Psi}_w^2\right) = \Psi_w \otimes * \Psi_w \quad ,$$

$$G_3 = \Psi_w \otimes * \Psi_w \otimes * \Psi_w \quad ,$$

$$.$$

$$.$$

$$.$$

$$G_n = \Psi_w \overbrace{\qquad\qquad}^{n-1 \text{ convolutions}} \Psi_w \quad . \tag{3.25}$$

In summary, it is clear that the determination of $\Psi_w(x,t)$, as given by equation (3.24), is the basic building block of the calculation. This is the focus of the effort of the next section.

## 4. MATHEMATICAL STRUCTURE OF $G_n(x,t)$

From equation (3.10) we can write

$$\bar{\Psi}_w^n = \big(g(k,s)v(k,s)\big)^n = \bar{g}_n\bar{v}_n \quad , \tag{4.1}$$

where

$$\bar{g}_n(k,s) = \left(\frac{w_2 - w_1}{w_2 + w_1}\right)^n \quad , \tag{4.2}$$

$$\bar{v}_n(k,s) = \exp(-2n\lambda_1 l) \quad . \tag{4.3}$$

Examination of equation (3.17) shows that the electric field at the surface can be considered as a sum of terms involving multiple round-trip reflections from the second layer. This can be seen for example by first examining the inverse Laplace transform of $\bar{v}_n(k,s)$.

We have

$$\begin{aligned}\hat{v}_n(k,t) &= \underline{L}^{-1}\bar{v}_n(k,s)\\ &= \underline{L}^{-1}\exp(-2nl\sqrt{s\mu\sigma_1 + k^2})\\ &= \underline{L}^{-1}\exp(-2nl\sqrt{\mu\sigma_1}\sqrt{s+\alpha_1}) \quad ,\end{aligned} \tag{4.4}$$

where

$$\alpha_1 = \frac{k^2}{\mu\sigma_1} \quad . \tag{4.5}$$

Using the formula

$$\underline{L}^{-1} f(s+\alpha_1) = e^{-\alpha_1 t} \underline{L}^{-1} f(s) \quad , \tag{4.6}$$

we obtain

$$\hat{v}_n(k,t) = e^{-\alpha_1 t} \theta_n(t) \quad , \tag{4.7}$$

where

$$\theta_n(t) = \frac{n}{2\sqrt{\pi}} \frac{\sqrt{T_R}}{t^{3/2}} \exp\left(\frac{-n^2 T_R}{4t}\right) \tag{4.8}$$

and

$$T_R = L^2 \mu \sigma_1 \ = \ \text{two-way diffusion time to second layer} \quad ,$$

$$L = 2l = \text{round-trip distance}$$

As observed from equation (4.8), for

$$\frac{T_R}{t} \quad > \quad 1 \tag{4.9}$$

the damping will become severe and perhaps only one term in the expansion will be necessary. Moreover, $\theta_n$ is in general a rapidly decreasing function of $n$, as can be seen by examining its maximum value, $\theta_{n,max}$. This is determined by solving the equation

$$\frac{d\theta_n}{dt} = 0 \tag{4.10}$$

for the time at which the maximum occurs; this time is given by

$$t_m = \frac{n^2 T_R}{6} \quad , \tag{4.11}$$

and the corresponding value of $\theta_n$ is

$$\theta_{n,max} = \frac{6^{3/2}}{2\sqrt{\pi}} \frac{1}{T_R} \frac{1}{n^2} e^{-\frac{3}{2}} \quad . \tag{4.12}$$

The foregoing equation supports the conjecture that higher-order terms provide diminishing contributions to the overall solution.

The main difficulty in determining

$$G_n(x,t) = \underline{L}^{-1}\,\underline{M}^{-1}\bar{\Psi}_w^n = \underline{M}^{-1}\underline{L}^{-1}\bar{\Psi}_w^n \tag{4.13}$$

is attributed to performing the inverse Laplace transform of $\bar{g}_n(k,s)$. Using equation (3.10) in equation (4.2) yields

$$\bar{g}_n(k,s) = \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)^n = \bar{g}_1^n \quad , \tag{4.14}$$

where

$$\bar{g}_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \tag{4.15}$$

and $\lambda_1, \lambda_2$ are given by equation (3.10). Let us first consider some of the mathematical properties of $\bar{g}_1^n$. We write $\bar{g}_1$ in the form

$$\bar{g}_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\,\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 - \lambda_2^2} \quad . \tag{4.16}$$

Using equation (3.10) we have

$$\lambda_1^2 - \lambda_2^2 = s\mu(\sigma_1 - \sigma_2) \quad . \tag{4.17}$$

Substituting equations (4.16) and (4.17) into equation (4.14), and subsequently using the binomial expansion for $(\lambda_1 - \lambda_2)^{2n}$ we obtain

$$\bar{g}_n = \left(\frac{1}{s\mu(\sigma_1 - \sigma_2)}\right)^n \sum_{m=0}^{2n} b_{2n,m}\lambda_1^{2n-m}(-\lambda_2)^m \quad , \tag{4.18}$$

where $b_{2n,m}$ is the binomial coefficient

$$b_{2n,m} = \frac{(2n)!}{(2n-m)!m!} \quad . \tag{4.19}$$

Let us now consider the structure of the terms in equation (4.18). For the even terms, characterized by

$$m = 2m' \quad , \quad 0 \leq m' \leq n \quad , \tag{4.20}$$

we have

$$\lambda_1^{2n-2m'}(-\lambda_2)^{2m'} = \left(s\mu\sigma_1 + k^2\right)^{n-m'}\left(s\mu\sigma_2 + k^2\right)^{m'} \quad . \tag{4.21}$$

For the odd numbered terms, characterized by

$$m = 2m'' + 1 \quad , \quad 0 \leq m'' \leq n - 1 \quad , \tag{4.22}$$

we can write

$$\lambda_1^{2n-m''}(-\lambda_2)^{m''} = -\left(s\mu\sigma_1 + k^2\right)^{n-m''-1}\left(s\mu\sigma_2 + k^2\right)^{m''} s\bar{\Gamma} \tag{4.23}$$

where

$$\bar{\Gamma} = \frac{\sqrt{s\mu\sigma_1 + k^2}\sqrt{s\mu\sigma_2 + k^2}}{s} \quad . \tag{4.24}$$

By expressing $\left(s\mu\sigma_1 + k^2\right)^{n-m'}$, $\left(s\mu\sigma_2 + k^2\right)^{m'}$, $\left(s\mu\sigma_1 + k^2\right)^{m''}$, and $\left(s\mu\sigma_2 + k^2\right)^{m''}$ as a binomial expansion, and substituting the results into equation (4.18), it can be shown by combining the summations that $\bar{g}_n$ is of the form

$$\bar{g}_n = \bar{g}_{n1} + \bar{g}_{n2} \quad ,$$

where

$$\bar{g}_{n1} = \frac{1}{s^n\left(\mu(\sigma_1 - \sigma_2)\right)^n}\left(\sum_{r=0}^{n} A_r s^r k^{2p}\right) \tag{4.25}$$

$$\bar{g}_{n2} = \frac{1}{s^n\left(\mu(\sigma_1 - \sigma_2)\right)^n}\left(\sum_{r=0}^{n} B_r s^r k^{2q}\right) s\bar{\Gamma} \quad . \tag{4.26}$$

In the foregoing expressions, $A_r$ and $B_r$ are constants which depend on $\sigma_1, \sigma_2$ and the index $r$, and the integers $p$ and $q$ are positive linear functions of the summation index, $r$.

It is not necessary to go into the tedious details to establish the implications of equations (4.25) and (4.26) regarding the computation of $G_n(x,t)$ from equation (4.13).

Using equations (4.1) and (4.3) in equations (4.25) and (4.26) we have

$$\bar{\Psi}^n_w = \bar{\Psi}^n_{w,1} + \bar{\Psi}^n_{w,2}$$

where

$$\bar{\Psi}^n_{w,1} = \frac{1}{(\mu(\sigma_1 - \sigma_2))^n} \sum_r^n A_r \left( \frac{k^{2p}}{s^{n-r}} \, \bar{v}_n(k,s) \right) \quad , \tag{4.27}$$

$$\bar{\Psi}^n_{w,2} = \frac{1}{(\mu(\sigma_1 - \sigma_2))^n} \sum_r^{n-1} B_r \left( \frac{k^{2q}}{s^{n-r}} \, \bar{v}_n(k,s) s \bar{\Gamma} \right) \quad . \tag{4.28}$$

From equation (4.13) we deduce

$$G_n(x,t) = G_{n1}(x,t) + G_{n2}(x,t) \quad , \tag{4.29}$$

where

$$G_{n1} = \frac{1}{(\mu(\sigma_1 - \sigma_2))^n} \sum_r^n A_r Y_r(x,t) \quad , \tag{4.30}$$

$$G_{n2} = \frac{1}{(\mu(\sigma_1 - \sigma_2))^n} \sum_r^{n-1} B_r Z_r(x,t) \quad , \tag{4.31}$$

$$Y_r(x,t) = \underline{M}^{-1} \, \underline{L}^{-1} \left( \frac{k^{2p}}{s^{n-r}} \, \bar{v}_n(k,s) \right) \quad , \tag{4.32}$$

$$Z_r(x,t) = \underline{M}^{-1} \, \underline{L}^{-1} \left( \frac{k^{2q}}{s^{n-r}} \, \bar{v}_n(k,s) s \bar{\Gamma} \right) \quad . \tag{4.33}$$

If we now let

$$v_n(x,t) = \underline{M}^{-1} \, \underline{L}^{-1} \bar{v}_n(k,s) \tag{4.34}$$

and

$$\Phi_n(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}(\bar{v}_n(k,s)s\bar{\Gamma}(k,s)) \quad , \tag{4.35}$$

it then follows from equations (2.6) and (2.7) that

$$Y_r(x,t) = \int \cdots \int dt_1\ dt_2 \cdots dt_{n-r}\left(-\frac{\partial^2}{\partial x^2}\right)^p v_n(x,t_1) \quad , \tag{4.36}$$

$$Z_r(x,t) = \int \cdots \int dt_1\ dt_2 \cdots dt_{n-r}\left(-\frac{\partial^2}{\partial x^2}\right)^q \Phi_n(x,t_1) \quad . \tag{4.37}$$

In summary, we have shown that if one can determine $v_n(x,t)$ and $\Phi_n(x,t)$, it is possible to determine the Green's function $G_n(x,t)$ by space derivates on $v_n$ and $\Phi_n$ followed by repeated time integrations. The utility of this approach depends on the simplicity and speed of performing the time integration.

In equation (4.7) we showed

$$\hat{v}_n(k,t) = \underline{L}^{-1}\bar{v}_n(k,s) = \exp\left(-\frac{k^2}{\mu\sigma_1}t\right)\theta_n(t) \quad . \tag{4.38}$$

We now have

$$v_n(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\bar{v}_n(k,s) = \theta_n(t)R(x,t) \quad , \tag{4.39}$$

where

$$\begin{aligned} R(x,t) &= \int_{-\infty}^{\infty} e^{ikx} \exp\left(-\frac{k^2 t}{\mu\sigma_1}\right)\, dk \\ &= \frac{\sqrt{\pi\mu\sigma_1}}{\sqrt{t}} \exp\left(-\frac{x^2\mu\sigma_1}{4t}\right) \quad . \end{aligned} \tag{4.40}$$

It is observed that $v_n(x,t)$ can be expressed in closed form, which facilitates the computation of equation (4.36).

Now let us consider the deduction of $\Phi_n(x,t)$. We have

$$\begin{aligned} \Phi_n(x,t) &= \underline{M}^{-1}\,\underline{L}^{-1}\left(s\bar{v}_n(k,s)\bar{\Gamma}(k,s)\right) \\ &= \frac{\partial}{\partial t}\left(\underline{M}^{-1}\,\underline{L}^{-1}\bar{P}_n(k,s)\right) \quad , \end{aligned} \tag{4.41}$$

where

$$\bar{P}_n(k,s) = \bar{v}_n(k,s)\bar{\Gamma}(k,s) \quad . \tag{4.42}$$

By examining equations (4.7), (4.8), and (4.24), which are the constitutents of equation (4.42), we notice that the only difference between $\bar{P}_n$ and $\bar{P}_1(k,s)$ is the replacement of $T_R$ by $n^2T_R$. Thus, if we can determine

$$P_1(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\bar{P}_1(k,s) \quad , \tag{4.43}$$

we can determine $P_n(x,t)$ by substituting $n^2T_R$ for $T_R$ in the resultant expression.

For the purposes of this investigation we are limiting the calculation to the evaluation of

$$G_1(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\big(\bar{g}_1(k,s)\bar{v}_1(k,s)\big) \quad . \tag{4.44}$$

This provides the contribution to the surface value of the electric field from the first round-trip reflection. In addition, section 5 shows that the calculation of $G_1(x,t)$ includes as one of its components the computation of $\bar{P}_1(x,t)$.

## 5. CALCULATION OF $G_1(x,t)$

Starting from equation (4.44) we have

$$\begin{aligned} G_1(x,t) &= \underline{M}^{-1}\,\underline{L}^{-1}\big(\bar{g}_1(k,s)\bar{v}_1(k,s)\big) \\ &= \underline{M}^{-1}\int_0^t \hat{g}_1(k,t')\hat{v}_1(k,t-t')\;dt' \\ &= \int_{-\infty}^{\infty} e^{ikx}\int_0^t \hat{g}_1(k,t')\hat{v}_1(k,t-t')\;dt'\;dk \quad . \end{aligned} \tag{5.1}$$

Using equation (4.16) we can write

$$\bar{g}_1 = \frac{1}{s\mu(\sigma_1-\sigma_2)}\left(\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\right) \quad , \tag{5.2}$$

$$\bar{g}_1 = \frac{s\mu(\sigma_1 + \sigma_2) + 2k^2 - 2\mu\sqrt{\sigma_1\sigma_2}\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s\mu(\sigma_1 - \sigma_2)} \quad , \tag{5.3}$$

where

$$\alpha_1 = \frac{k^2}{\mu\sigma_1} \quad ,$$
$$\alpha_2 = \frac{k^2}{\mu\sigma_2} \quad . \tag{5.4}$$

We also have

$$\hat{g}_1(k,t) = \frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2}\delta(t) + \frac{2k^2}{\mu(\sigma_1 - \sigma_2)}H(t) - \frac{2\sqrt{\sigma_1\sigma_2}}{\sigma_1 - \sigma_2}\underline{L}^{-1}\bar{Q} \tag{5.5}$$

where

$$\bar{Q} = \frac{\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s} = \frac{1}{\mu\sqrt{\sigma_1\sigma_2}}\,\bar{\Gamma} \quad ,$$
$$\delta(t) = \text{delta function} \quad ,$$
$$H(t) = \text{step function} \quad . \tag{5.6}$$

From equation (5.5) we can write

$$\hat{g}_1(k,t) = \hat{g}_{11} + \hat{g}_{12} + \hat{g}_{13} \quad , \tag{5.7}$$

where

$$\hat{g}_{11} = \frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2}\ \delta(t) \quad ,$$
$$\hat{g}_{12} = \frac{2k^2}{\mu(\sigma_1 - \sigma_2)}H(t) \quad ,$$
$$\hat{g}_{13} = \frac{-2\sqrt{\sigma_1\sigma_2}}{\sigma_1 - \sigma_2}\ \hat{Q}(k,t) \quad ,$$
$$\hat{Q}(k,t) = \underline{L}^{-1}\left(\frac{\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s}\right) \quad . \tag{5.8}$$

As shown in appendix A, $\hat{Q}(k,t)$ is given by

$$\hat{Q}(k,t) = e^{-\alpha t}\left[\beta I_1(\beta t) + \alpha I_o(\beta t)\right]$$
$$+ (\alpha^2 - \beta^2)\int_0^t e^{-\alpha u} I_o(\beta u)\ du + \delta(t) \quad , \tag{5.9}$$

where

$$\alpha = \frac{1}{2\mu}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)k^2 \quad ,$$
$$\beta = \frac{1}{2\mu}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)k^2 \quad . \tag{5.10}$$

Using equations (4.7) and (4.8) we can write

$$\hat{v}_1(k,t) = \exp\left(-\frac{k^2 t}{\mu\sigma_1}\right)\left(\frac{\sqrt{T_R}}{2\sqrt{\pi}t^{3/2}}\right)\exp\left(-\frac{T_R}{4t}\right) \tag{5.11}$$

If we insert equations (5.7) to (5.11) into equation (5.1), and note that we can combine the $\delta(t)$ of equation (5.9) with that of equation (5.8), we can write

$$G_1(x,t) = G_{1,1} + G_{1,2} + G_{1,3} \quad , \tag{5.12}$$

where

$$G_{1,j} = \int_{-\infty}^{\infty} e^{ikx} \int_0^t \hat{\phi}_j(k,t')\hat{v}_1(k,t-t')\, dt'\, dk \quad , \quad j = 1,2,3 \quad , \tag{5.13}$$

$$\hat{\phi}_1(k,t) = \frac{\sqrt{\sigma_1} - \sqrt{\sigma_2}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}}\ \delta(t) \quad , \tag{5.14}$$

$$\hat{\phi}_2(k,t) = \frac{2k^2}{\mu(\sigma_1 - \sigma_2)}\ H(t) \quad , \tag{5.15}$$

$$\hat{\phi}_3(k,t) = \frac{-2\sqrt{\sigma_1\sigma_2}}{\sigma_1 - \sigma_2}\left[\hat{L}_1(k,t) + \hat{L}_2(k,t) + \hat{L}_3(k,t)\right] \quad , \tag{5.16}$$

$$\hat{L}_1(k,t) = \beta e^{-\alpha t} I_1(\beta t) \quad , \tag{5.17}$$

$$\hat{L}_2(k,t) = \alpha e^{-\alpha t} I_o(\beta t) \quad , \tag{5.18}$$

$$\hat{L}_3(k,t) = (\alpha^2 - \beta^2)\int_0^t e^{-\alpha u} I_o(\beta u)\, du \quad . \tag{5.19}$$

It should be noted that although $\sigma_1 - \sigma_2$ appears in the denominator of equations (5.15) and (5.16), the sum of $\phi_2 + \phi_3$ must equal zero in the limit of $\sigma_1 = \sigma_2$ (note that $\phi_1$ is already zero in this case). This follows from the observation that there can be no reflections in this situation. This is easily seen by using approximations to $\hat{L}_1, \hat{L}_2,$ and $\hat{L}_3$ in the limit of $\beta \to 0$.

In the next subsections we perform the calculations of $G_{1,j}$.

### 5.1 Calculation of $G_{1,1}(x,t)$

From equations (5.13) and (5.14) we have

$$\begin{aligned}\int_0^t \hat{\phi}_1(k,t) v_1(k,t-t')\,dt' &= \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}} \int_0^t \delta(t')\hat{v}_1(k,t-t')\,dt' \\ &= \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}} \exp\left(-\frac{k^2 t}{\mu\sigma_1}\right) f(t) \end{aligned} \tag{5.20}$$

where

$$f(t) = \frac{\sqrt{T_R}}{2\sqrt{\pi} t^{3/2}} \exp\left(\frac{-T_R}{4t}\right) \quad . \tag{5.21}$$

Performing the integration over $k$ space yields

$$\begin{aligned} G_{1,1} &= \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}} f(t) \int_{-\infty}^{\infty} e^{ikx} \exp\left(-\frac{k^t}{\mu\sigma_1}\right) dk \\ &= \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}} f(t) \exp\left(-\frac{x^2\mu\sigma_1}{4t}\right) \frac{\sqrt{\pi\mu\sigma_1}}{\sqrt{t}} \\ &= \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}} \frac{1}{2} \frac{T_R}{t^2} \frac{1}{L} \exp\left[-\frac{T_R}{4t}\left(1+\frac{x^2}{L^2}\right)\right] \quad . \end{aligned} \tag{5.22}$$

### 5.2 Calculation of $G_{1,2}$

From equations (5.13) and (5.15) we have

$$G_{1,2} = \int_{-\infty}^{\infty} e^{ikx} \int_0^t \frac{2k^2}{\mu(\sigma_1 - \sigma_2)} \exp\left(-\frac{k^2 t'}{\mu\sigma_1}\right) f(t')\,dt'\,dk \quad . \tag{5.23}$$

Replacing $k^2$ by $-\partial^2/\partial x^2$ we obtain

$$G_{1,2} = -\frac{2}{\mu(\sigma_1 - \sigma_2)} \frac{\partial^2}{\partial x^2} \int_0^t f(t')\ dt' \int_{-\infty}^{\infty} e^{ikx} \exp\left(\frac{k^2 t'}{\mu \sigma_1}\right)\ dk \tag{5.24}$$

The integral in equation (5.24) is the same as that in equation (5.22), so that we can write

$$G_{1,2} = -\frac{2}{\mu(\sigma_1 - \sigma_2)} \frac{\partial^2}{\partial x^2} \int_0^t \frac{1}{2} \frac{T_R}{(t')^2} \frac{1}{L} \exp\left(-\frac{1}{4} \frac{T_R(1+\frac{x^2}{L^2})}{t'}\right) dt' \quad . \tag{5.25}$$

Making the substitution

$$u' = \frac{1}{4} \frac{T_R}{t'} \left(1 + \frac{x^2}{L^2}\right) = \frac{r}{t'} \tag{5.26}$$

permits the integration to be performed through the relationship

$$du' = -\frac{r}{t'^2}\ dt' \quad . \tag{5.27}$$

We obtain

$$G_{1,2}(x,t) = -\frac{4}{\mu L} \frac{1}{\sigma_1 - \sigma_2} \frac{\partial^2}{\partial x^2} U(x,t) \quad , \tag{5.28}$$

where

$$U(x,t) = \frac{1}{1+\frac{x^2}{L^2}} \exp\left(-\frac{T_R}{4} \frac{(1+\frac{x^2}{L^2})}{t}\right) \quad . \tag{5.29}$$

Using equation (3.19), the contribution from the first reflection will be given by

$$E_{y,1} = G_1(x,t) \otimes * E_{y,0}(x,t) \quad , \tag{5.30}$$

and in particular the contribution from $G_{1,2}$ is

$$E_{y,1,2} = \frac{-4}{\pi L} \frac{1}{\sigma_1 - \sigma_2} \int_{-\infty}^{\infty} \int_0^t \left(\frac{\partial^2}{\partial x'^2} U(x',t')\right) E_{y,0}(x-x', t-t')\ dx'\ dt' \quad . \tag{5.31}$$

Based on application of the Faltung theorem discussed in appendix B, the foregoing integration can be converted to the form

$$E_{y,1,2} = \frac{-4}{\pi L}\frac{1}{\sigma_1 - \sigma_2}\int_{-\infty}^{\infty}\int_0^t U(x',t')\left(\frac{\partial^2 E_{y,0}(z,t-t')}{\partial z^2}\right)_{z=x-x'} dx'\, dt' \quad . \tag{5.32}$$

It may turn out that for computational purposes the form given by equation (5.32) is easier to evaluate.

## 5.3 Calculation of $G_{1,3}$

Inserting equation (5.16) into equation (5.13) and rearranging gives the following expression for $G_{1,3}$:

$$G_{1,3}(x,t) = -\frac{2\sqrt{\sigma_1\sigma_2}}{\sigma_1 - \sigma_2}\int_0^t dt'\ f(t-t')\big(N_1(x,t') + N_2(x,t') + N_3(x,t')\big) \quad , \tag{5.33}$$

where

$$N_1 = \int_{-\infty}^{\infty} \beta e^{ikx} e^{-\alpha t'} I_1(\beta t') \exp\left(-\frac{k^2}{\mu\sigma_1}(t-t')\right)\ dk \quad , \tag{5.34}$$

$$N_2 = \int_{-\infty}^{\infty} \alpha e^{ikx} e^{-\alpha t'} I_o(\beta t') \exp\left(-\frac{k^2}{\mu\sigma_1}(t-t')\right)\ dk \quad , \tag{5.35}$$

$$N_3 = \int_{-\infty}^{\infty} (\alpha^2 - \beta^2) e^{ikx} \int_0^{t'} e^{-\alpha u} I_o(\beta u)\ du\ \exp\left(-\frac{k^2}{\mu\sigma_1}(t-t')\right)\ dk \quad . \tag{5.36}$$

equations (5.34) to (5.36) can be simplified using the formulas

$$I_o(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} d\theta \quad , \tag{5.37}$$

$$I_1(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos\theta\ d\theta \quad , \tag{5.38}$$

and writing $\alpha$ and $\beta$ in the form

$$\alpha = \frac{1}{\mu\sigma_a}k^2 \quad , \tag{5.39}$$

$$\beta = \frac{1}{\mu\sigma_b}k^2 \quad , \tag{5.40}$$

where

$$\frac{1}{\sigma_a} = \frac{1}{2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) \quad , \quad \sigma_a = \frac{2(\sigma_1\sigma_2)}{\sigma_2 + \sigma_1} \quad , \tag{5.41}$$

$$\frac{1}{\sigma_b} = \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) \quad , \quad \sigma_b = \frac{2(\sigma_1\sigma_2)}{\sigma_2 - \sigma_1} \quad . \tag{5.42}$$

Using equations (5.37) to (5.42) in equations (5.34) and (5.35) yields

$$\begin{aligned} N_1 &= -\frac{\partial^2}{\partial x^2}\, n_1 \quad , \\ N_2 &= -\frac{\partial^2}{\partial x^2}\, n_2 \quad , \end{aligned} \tag{5.43}$$

where

$$\begin{aligned} n_1 = \frac{1}{\mu\sigma_b\pi}\int_0^\pi \cos\theta\ d\theta \int_{-\infty}^{\infty} e^{ikx}\exp\left(-(\frac{1}{\mu\sigma_a})k^2t'\right) \\ \times\ \exp\left((\frac{1}{\mu\sigma_b})k^2t'\cos\theta\right)\exp\left(-\frac{k^2}{\mu\sigma_1}(t-t')\right)\ dk \quad , \end{aligned} \tag{5.44}$$

$$\begin{aligned} n_2 = \frac{1}{\mu\sigma_a\pi}\int_0^\pi d\theta \int_{-\infty}^{\infty} e^{ikx}\exp\left(-(\frac{1}{\mu\sigma_a})k^2t'\right) \\ \times\ \exp\left((\frac{1}{\mu\sigma_b})k^2t'\cos\theta\right)\exp\left(-\frac{k^2}{\mu\sigma_1}(t-t')\right)\ dk \quad . \end{aligned} \tag{5.45}$$

The expression for $N_3$ is simplified by first performing the integration over $u$ in equation (5.36). We have

$$\begin{aligned} \int_0^{t'} e^{-\alpha u}I_o(\beta u)\ du &= \frac{1}{\pi}\int_0^\pi d\theta \int_0^{t'} e^{-\alpha u}e^{\beta u\cos\theta}\ du \\ &= \frac{1}{\pi}\int_0^\pi d\theta \frac{1 - e^{-(\alpha-\beta\cos\theta)t'}}{\alpha - \beta\cos\ \theta} \quad . \end{aligned} \tag{5.46}$$

Inserting equation (5.46) into equation(5.36) yields

$$N_3 = -\frac{\partial^2}{\partial x^2} n_3 \quad , \tag{5.47}$$

where

$$\begin{aligned} n_3 = \frac{1}{\mu \sigma_a \pi} \int_0^\pi d\theta \; S(\theta) \int_{-\infty}^{\infty} e^{ikx} \exp\left( -\frac{k^2 (t-t')}{\mu \sigma_1} \right) \\ \times \left( 1 - \exp\left( -(\frac{1}{\mu \sigma_a} - \frac{\cos\theta}{\mu \sigma_b}) k^2 t' \right) \right) dk \quad , \end{aligned} \tag{5.48}$$

$$S(\theta) = \frac{1 - (\frac{\sigma_a}{\sigma_b})^2}{1 - (\frac{\sigma_a}{\sigma_b}) \cos \; \theta} \quad . \tag{5.49}$$

Using the general formula

$$\int_{-\infty}^{\infty} e^{ikx} e^{-\Lambda k^2} dk = \frac{\sqrt{\pi}}{\sqrt{\Lambda}} \exp\left( \frac{-x^2}{4\Lambda} \right) = F(\Lambda) \quad , \tag{5.50}$$

with

$$\Lambda_1 = \frac{1}{\mu \sigma_a} \left( 1 - \frac{\sigma_a}{\sigma_b} \cos\theta \right) t' + \frac{1}{\mu \sigma_1} (t - t') \quad , \tag{5.51}$$

$$\Lambda_2 = \frac{t - t'}{\mu \sigma_1} \quad , \tag{5.52}$$

we deduce

$$n_3 = \frac{1}{\mu \sigma_a \pi} C' F(\Lambda_2) - \frac{1}{\mu \sigma_a \pi} \int_0^\pi d\theta \; S(\theta) F(\Lambda_1) \quad , \tag{5.53}$$

where

$$C' = \int_0^\pi S(\theta) \; d\theta \quad . \tag{5.54}$$

Noting that

$$\left| \frac{\sigma_a}{\sigma_b} \right| = \left| \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1} \right| < 1 \tag{5.55}$$

gives

$$\int_0^\pi \frac{d\theta}{1-(\sigma_a/\sigma_b)\cos\theta} = \frac{\pi}{\sqrt{1-(\sigma_a/\sigma_b)^2}} \tag{5.56}$$

and

$$C' = \pi\sqrt{1-(\sigma_a/\sigma_b)^2} \tag{5.57}$$

By comparing equation (5.51) with the exponential terms of equations (5.44) and (5.45) we obtain

$$n_1 = \frac{1}{\mu\sigma_b\pi}\int_0^\pi \cos\ \theta\ d\theta\ F(\Lambda_1) \quad , \tag{5.58}$$

$$n_2 = \frac{1}{\mu\sigma_a\pi}\int_0^\pi d\theta\ F(\Lambda_2) \quad . \tag{5.59}$$

From equations (5.53), (5.58), and (5.59) we have

$$n = n_1 + n_2 + n_3 \quad , \tag{5.60}$$

$$n = \frac{\sqrt{1-(\sigma_a/\sigma_b)^2}}{\mu\sigma_a}F(\Lambda_2) + \frac{1}{\mu\sigma_a\pi}\int_0^\pi W(\theta)F(\Lambda_1)\ d\theta \quad , \tag{5.61}$$

where

$$\begin{aligned} W(\theta) &= \frac{\sigma_a}{\sigma_b}\cos\ \theta + 1 - S(\theta) \quad , \\ W(\theta) &= \frac{(\sigma_a/\sigma_b)^2\sin^2\theta}{1-(\sigma_a/\sigma_b)\cos\ \theta} \quad . \end{aligned} \tag{5.62}$$

Combining equations (5.43), (5.47), (5.60), (5.61) and inserting the latter into (5.33) gives

$$G_{1,3} = \frac{\partial^2}{\partial x^2}\left[ABI_1 + ACI_2\right] \quad , \tag{5.63}$$

where

$$
\begin{aligned}
A &= \frac{2\sqrt{\sigma_1\sigma_2}}{\sigma_1 - \sigma_2} \\
B &= \frac{\sqrt{1 - (\sigma_a/\sigma_b)^2}}{\mu\sigma_a} = \frac{\sqrt{1-\rho^2}}{\mu\sigma_a} \\
C &= \frac{(\sigma_a/\sigma_b)^2}{\mu\sigma_a\pi} = \frac{\rho^2}{\mu\sigma_a\pi} \\
I_1 &= \int_0^t dt'\ f(t-t')F(\Lambda_2) \\
I_2 &= \int_0^t dt'\ f(t-t')\langle F(\Lambda_1)\rangle \\
\langle F(\Lambda_1)\rangle &= \int_0^\pi \frac{\sin^2\theta}{1-\rho\cos\ \theta}F(\Lambda_1) \\
\rho &= \frac{\sigma_a}{\sigma_b} = \frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}
\end{aligned} \tag{5.64}
$$

It is easy to see that $I_1$ reduces to the same integral as that of equation (5.25); that is,

$$
I_1 = \frac{2}{L}\,U(x,t) \quad , \tag{5.65}
$$

where $U(x,t)$ is given by equation (5.29).

Using equations (5.41) and (5.42) we deduce

$$
\frac{\partial^2(ABI_1)}{\partial x^2} = \frac{4}{\mu L(\sigma_1-\sigma_2)}\frac{\partial^2}{\partial x^2}U(x,t) = -G_{1,2}(x,t) \quad . \tag{5.66}
$$

We observe that the foregoing contribution provides an *exact cancellation* to $G_{1,2}$.

Thus, the significant contribution to $G_{1,3}$ comes from the term $ACI_2$. Unfortunately, this does not appear to be easily integrated, and numerical techniques must be used. For simplicity in presentation we shall render the results in dimensionless form. We first compute the constant $AC$; we have

$$
AC = \frac{1}{\mu\sqrt{\sigma_1\sigma_2}}\,\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2} = \frac{K}{\mu\sigma_1} \tag{5.67}
$$

where

$$K = \sqrt{\frac{\sigma_1}{\sigma_2}} \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \tag{5.68}$$

Let us now introduce the variable

$$y = \frac{t'}{t} \tag{5.69}$$

in equation (5.64d). We can write $I_2$ in terms of $y$ in the following form:

$$I_2 = \frac{1}{2L} \Pi_o(t_d, \phi_d) \quad , \tag{5.70}$$

where

$$\begin{aligned}
\Pi_o &= \frac{1}{t_d} \int_0^1 \frac{1}{(1-y)^{3/2}} \exp\left(-\frac{1}{4t_d(1-y)}\right) \Omega(\phi_d, y)\, dy \quad , \\
\Omega(\phi_d, y) &= \int_0^\pi \frac{\sin^2\theta\, d\theta}{1 - \rho\cos\theta} \frac{1}{\sqrt{\gamma_o}} \exp\left(-\frac{\phi_d}{\gamma_o}\right) \quad , \\
\gamma_o &= 1 + \left(\frac{1-r}{2r}\right) y(1 + \cos\theta) \quad , \\
\phi_d &= \frac{x_d^2}{4t_d} \quad , \\
t_d &= \frac{t}{T_R} \quad , \\
x_d &= \frac{x}{L} \quad , \\
r &= \frac{\sigma_2}{\sigma_1} \quad , \\
\rho &= \frac{r-1}{r+1} \quad .
\end{aligned} \tag{5.71}$$

Using the dimensionless variables introduced in equation (5.71) we write, for the Green's function,

$$G_1 = G_{1,1} + \frac{\partial^2}{\partial x^2}(ACI_2) \quad , \tag{5.72}$$

$$
G_1 = \frac{1}{2}\,\frac{1-\sqrt{r}}{1+\sqrt{r}}\,\frac{1}{T_R L}\,\frac{1}{t_d^2}\exp\left(-\frac{1+x_d^2}{4t_d}\right)
+ \frac{1}{2}\,\frac{1}{\sqrt{r}}\,\frac{1-r}{1-r}\,\frac{1}{T_R L}\,\frac{\partial^2}{\partial x_d^2}\,\Pi_o \quad . \tag{5.73}
$$

The first term in equation (5.73) is that of $G_{1,1}$ (cf. equation (5.22c)), expressed in terms of $T_R, L$, and the dimensionless variables $r$, $t_d$, and $x_d$.

The contribution to the electric field from $G_1$ is given from equation (5.30), where the required integrations can be performed in either $x_d/t_d$ or $x/t$ space. It is also evident from equations (5.31) and (5.32) that the term involving $\partial^2\Pi_o/\partial x_d^2$ can be converted to an integration of $\Pi_o$ combined with the second partial derivative of $E_{y,o}$.

Since $\Pi_o$ cannot be expressed in closed form, it must be provided in tabular form, either as a function of $t_d, x_d$ or $t_d, \phi_d$. This consideration should be reserved when the actual implementation of this result is incorporated in the HABEMP finite-difference model.

Tabular values of $\Pi_o(t_d, \phi_d)$ for selected values of $t_d$ and $\phi_d$ are rendered in table 1 (pp 30 and 31). Plots of $\Pi_o$ as a function of $\phi_d$ with $t_d$ as a parameter for $r = 0.5$ are shown in figure 3, while plots of $\Pi_o$ as a function of $t_d$ with $\phi_d$ as a parameter for $r = 0.5$ are shown in figure 4.

## 6. CONCLUSION

A solution for the two region, two-dimensional electromagnetic ground response has been developed which relates the surface components of the electric field to the surface components of the magnetic field. This has been accomplished by deriving a universal functional form for a dimensionless Green's function. The Green's function provides increasingly more accurate approximations to the response for each successive reflection from the second layer. This result would appear to provide simplification and reduced computer running time in the numerical modelling of the HABEMP when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem.

## ACKNOWLEDGEMENT

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Table 1. Values of $\Pi_O(t_d, \phi_d)$ for $r = 0.3, 0.5, 0.7, 2.0$, and $5.0$

$r = 0.3$

| | $\phi_d$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 | 3.00 | 4.00 | 5.00 |
| 0.10 | 0.4189 | 0.3450 | 0.2847 | 0.2354 | 0.1949 | 0.1345 | 0.0935 | 0.0463 | 0.0237 | 0.0126 |
| 0.30 | 1.7276 | 1.4584 | 1.2341 | 1.0469 | 0.8902 | 0.6485 | 0.4771 | 0.2657 | 0.1533 | 0.0915 |
| 0.50 | 2.0805 | 1.7740 | 1.5166 | 1.2999 | 1.1171 | 0.8312 | 0.6247 | 0.3630 | 0.2184 | 0.1355 |
| 0.75 | 2.1579 | 1.8532 | 1.5957 | 1.3776 | 1.1925 | 0.9003 | 0.6865 | 0.4104 | 0.2537 | 0.1615 |
| 1.00 | 2.1258 | 1.8340 | 1.5865 | 1.3760 | 1.1965 | 0.9117 | 0.7015 | 0.4268 | 0.2682 | 0.1734 |
| 1.50 | 1.9976 | 1.7334 | 1.5081 | 1.3156 | 1.1506 | 0.8867 | 0.6899 | 0.4288 | 0.2750 | 0.1810 |
| 2.00 | 1.8634 | 1.6227 | 1.4169 | 1.2459 | 1.0951 | 0.9649 | 0.7545 | 0.5954 | 0.3801 | 0.2497 |
| 3.00 | 1.6254 | 1.4212 | 1.2459 | 1.0951 | 0.9649 | 0.7545 | 0.5954 | 0.3801 | 0.2497 | 0.1681 |
| 4.00 | 1.4368 | 1.2592 | 1.1065 | 0.9747 | 0.8607 | 0.6760 | 0.5356 | 0.3446 | 0.2280 | 0.1544 |
| 5.00 | 1.2952 | 1.1370 | 1.0006 | 0.8828 | 0.7808 | 0.6151 | 0.4888 | 0.3161 | 0.2102 | 0.1430 |

$r = 0.5$

| | $\phi_d$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 | 3.00 | 4.00 | 5.00 |
| 0.10 | 0.4091 | 0.3340 | 0.2730 | 0.2235 | 0.1832 | 0.1236 | 0.0839 | 0.0393 | 0.0189 | 0.0093 |
| 0.30 | 1.7094 | 1.4274 | 1.1942 | 1.0010 | 0.8407 | 0.5965 | 0.4266 | 0.2233 | 0.1205 | 0.0669 |
| 0.50 | 2.0696 | 1.7447 | 1.4738 | 1.2476 | 1.0584 | 0.7664 | 0.5595 | 0.3054 | 0.1718 | 0.0993 |
| 0.75 | 2.1547 | 1.8288 | 1.5556 | 1.3260 | 1.1328 | 0.8319 | 0.6160 | 0.3459 | 0.2000 | 0.1187 |
| 1.00 | 2.1278 | 1.8140 | 1.5498 | 1.3270 | 1.1387 | 0.8438 | 0.6305 | 0.3603 | 0.2119 | 0.1278 |
| 1.50 | 2.0055 | 1.7193 | 1.4772 | 1.2720 | 1.0977 | 0.8226 | 0.6215 | 0.3630 | 0.2179 | 0.1340 |
| 2.00 | 1.8744 | 1.6125 | 1.3902 | 1.2012 | 1.0401 | 0.7848 | 0.5969 | 0.3531 | 0.2146 | 0.1335 |
| 3.00 | 1.6385 | 1.4151 | 1.2249 | 1.0625 | 0.9236 | 0.7022 | 0.5381 | 0.3229 | 0.1989 | 0.1253 |
| 4.00 | 1.4502 | 1.2553 | 1.0890 | 0.9467 | 0.8248 | 0.6298 | 0.4846 | 0.2931 | 0.1818 | 0.1153 |
| 5.00 | 1.3084 | 1.1343 | 0.9856 | 0.8582 | 0.7488 | 0.5735 | 0.4426 | 0.2692 | 0.1679 | 0.1069 |

$r = 0.7$

| | $\phi_d$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 | 3.00 | 4.00 | 5.00 |
| 0.10 | 0.4063 | 0.3301 | 0.2685 | 0.2187 | 0.1783 | 0.1189 | 0.0797 | 0.0363 | 0.0169 | 0.0080 |
| 0.30 | 1.7093 | 1.4189 | 1.1797 | 0.9824 | 0.8195 | 0.5732 | 0.4036 | 0.2042 | 0.1061 | 0.0565 |
| 0.50 | 2.0754 | 1.7385 | 1.4590 | 1.2266 | 1.0331 | 0.7368 | 0.5292 | 0.2789 | 0.1509 | 0.0837 |
| 0.75 | 2.1651 | 1.8257 | 1.5423 | 1.3054 | 1.1069 | 0.8004 | 0.5829 | 0.3158 | 0.1757 | 0.1001 |
| 1.00 | 2.1409 | 1.8130 | 1.5382 | 1.3075 | 1.1136 | 0.8123 | 0.5968 | 0.3290 | 0.1862 | 0.1078 |
| 1.50 | 2.0213 | 1.7210 | 1.4682 | 1.2549 | 1.0746 | 0.7925 | 0.5887 | 0.3316 | 0.1915 | 0.1131 |
| 2.00 | 1.8911 | 1.6155 | 1.3828 | 1.1859 | 1.0190 | 0.7565 | 0.5657 | 0.3227 | 0.1887 | 0.1127 |
| 3.00 | 1.6550 | 1.4193 | 1.2196 | 1.0500 | 0.9056 | 0.6774 | 0.5103 | 0.2953 | 0.1750 | 0.1059 |
| 4.00 | 1.4658 | 1.2598 | 1.0849 | 0.9360 | 0.8091 | 0.6078 | 0.4597 | 0.2682 | 0.1601 | 0.0975 |
| 5.00 | 1.3231 | 1.1389 | 0.9823 | 0.8488 | 0.7348 | 0.5536 | 0.4200 | 0.2463 | 0.1478 | 0.0905 |

Table 1. Values of $\Pi_0(t_d, \phi_d)$ for $r = 0.3, 0.5, 0.7, 2.0$, and $5.0$
(continued)

$r = 2.0$

| | $\phi_d$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 | 3.00 | 4.00 | 5.00 |
| 0.10 | 0.4258 | 0.3428 | 0.2762 | 0.2227 | 0.1797 | 0.1172 | 0.0767 | 0.0331 | 0.0145 | 0.0064 |
| 0.30 | 1.8148 | 1.4885 | 1.2223 | 1.0047 | 0.8268 | 0.5618 | 0.3835 | 0.1811 | 0.0871 | 0.0427 |
| 0.50 | 2.2159 | 1.8324 | 1.5171 | 1.2577 | 1.0439 | 0.7218 | 0.5015 | 0.2457 | 0.1227 | 0.0624 |
| 0.75 | 2.3210 | 1.9308 | 1.6083 | 1.3413 | 1.1202 | 0.7842 | 0.5519 | 0.2774 | 0.1421 | 0.0741 |
| 1.00 | 2.3011 | 1.9217 | 1.6070 | 1.3456 | 1.1282 | 0.7962 | 0.5649 | 0.2886 | 0.1502 | 0.0796 |
| 1.50 | 2.1798 | 1.8295 | 1.5376 | 1.2940 | 1.0905 | 0.7775 | 0.5572 | 0.2905 | 0.1543 | 0.0833 |
| 2.00 | 2.0436 | 1.7205 | 1.4505 | 1.2245 | 1.0351 | 0.7426 | 0.5355 | 0.2826 | 0.1519 | 0.0830 |
| 3.00 | 1.7928 | 1.5147 | 1.2816 | 1.0858 | 0.9211 | 0.6655 | 0.4833 | 0.2585 | 0.1408 | 0.0779 |
| 4.00 | 1.5900 | 1.3461 | 1.1412 | 0.9688 | 0.8236 | 0.5974 | 0.4355 | 0.2348 | 0.1288 | 0.0717 |
| 5.00 | 1.4365 | 1.2180 | 1.0341 | 0.8791 | 0.7484 | 0.5444 | 0.3980 | 0.2157 | 0.1189 | 0.0665 |

$r = 5.0$

| | $\phi_d$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 | 3.00 | 4.00 | 5.00 |
| 0.10 | 0.4777 | 0.3835 | 0.3080 | 0.2476 | 0.1991 | 0.1289 | 0.0837 | 0.0356 | 0.0153 | 0.0066 |
| 0.30 | 2.0449 | 1.6705 | 1.3660 | 1.1179 | 0.9158 | 0.6163 | 0.4162 | 0.1920 | 0.0899 | 0.0427 |
| 0.50 | 2.5014 | 2.0595 | 1.6974 | 1.4003 | 1.1565 | 0.7912 | 0.5435 | 0.2596 | 0.1259 | 0.0620 |
| 0.75 | 2.6237 | 2.1725 | 1.8008 | 1.4944 | 1.2414 | 0.8595 | 0.5975 | 0.2924 | 0.1454 | 0.0734 |
| 1.00 | 2.6034 | 2.1638 | 1.8004 | 1.4997 | 1.3506 | 0.8735 | 0.6114 | 0.3039 | 0.1535 | 0.0786 |
| 1.50 | 2.4690 | 2.0620 | 1.7240 | 1.4431 | 1.2093 | 0.8520 | 0.6029 | 0.3056 | 0.1573 | 0.0821 |
| 2.00 | 2.3163 | 1.9403 | 1.6272 | 1.3661 | 1.1482 | 0.8139 | 0.5793 | 0.2971 | 0.1547 | 0.0817 |
| 3.00 | 2.0366 | 1.7094 | 1.4385 | 1.2119 | 1.0221 | 0.7295 | 0.5228 | 0.2717 | 0.1433 | 0.0766 |
| 4.00 | 1.8043 | 1.5197 | 1.2814 | 1.0817 | 0.9141 | 0.6548 | 0.4711 | 0.2466 | 0.1310 | 0.0705 |
| 5.00 | 1.6307 | 1.3754 | 1.1613 | 0.9817 | 0.8307 | 0.5968 | 0.4304 | 0.2265 | 0.1209 | 0.0654 |

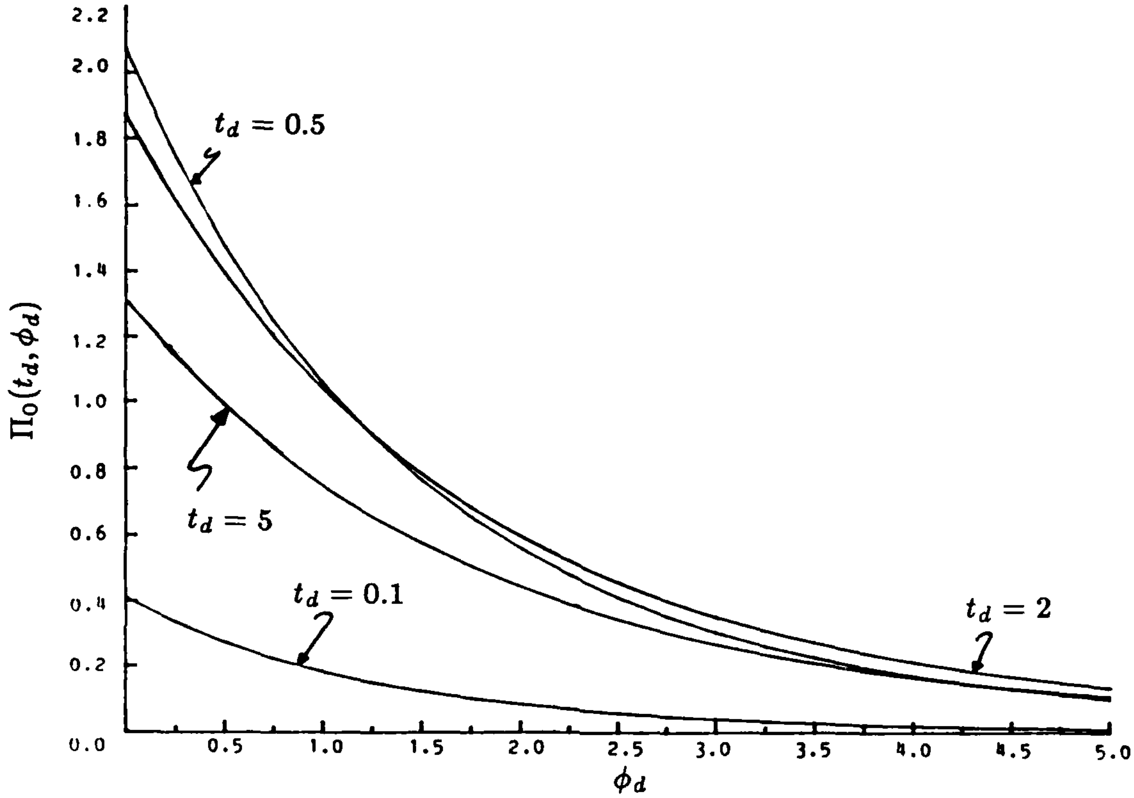


Figure 3. $\Pi_0(t_d, \phi_d)$ as a function of $\phi_d$ with $t_d$ as parameter for $r = 0.5$.

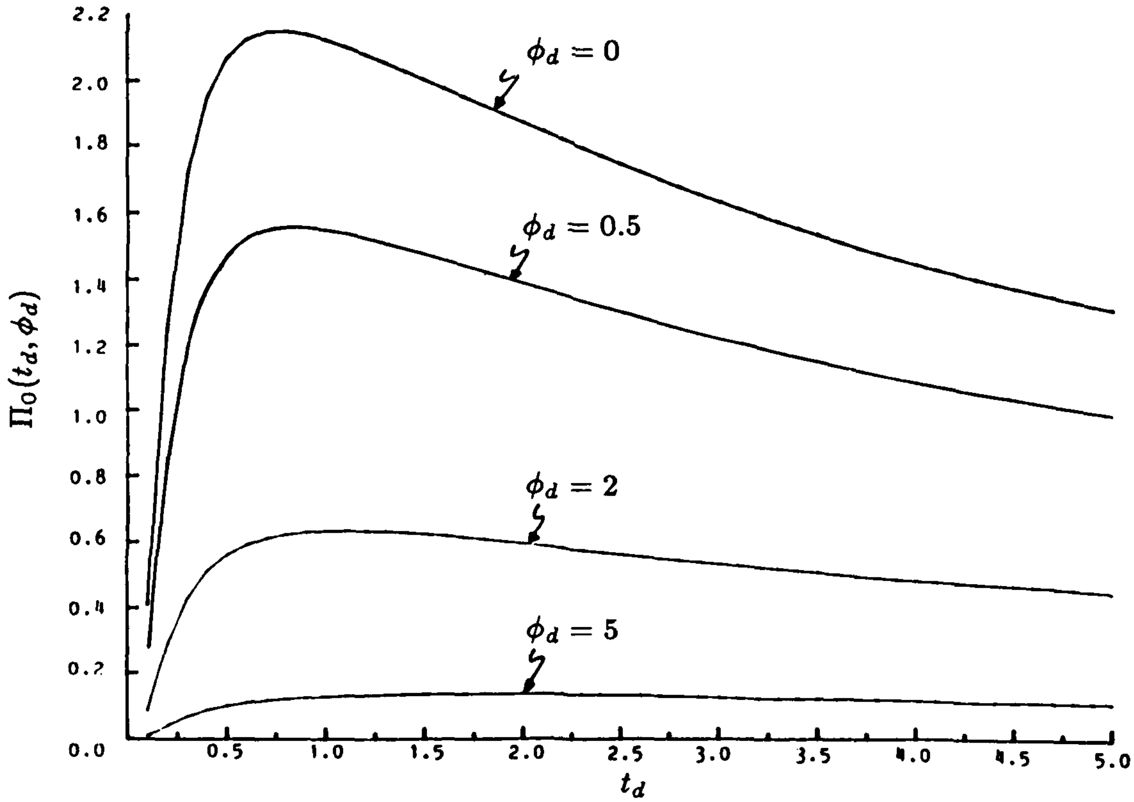


Figure 4. $\Pi_0(t_d, \phi_d)$ as a function of $t_d$ with $\phi_d$ as parameter for $r = 0.5$.

## APPENDIX A: CALCULATION OF $\hat{Q}(k,t)$

The basic building block for the calculation of

$$\hat{Q}(k,t) = \underline{L}^{-1}\left(\frac{\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s}\right) = \underline{L}^{-1}\left(\bar{Q}(k,s)\right) \tag{A.1}$$

is the formula

$$\begin{aligned}\underline{L}^{-1}\left(\bar{R}(s)\right) &= \underline{L}^{-1}\left[\frac{1}{s}\,\frac{(s+\alpha-\beta)^{\frac{1}{2}}}{(s+\alpha+\beta)^{\frac{1}{2}}}\right] = \hat{R}(t) \\ &= \left[e^{-\alpha t}I_o(\beta t) + (\alpha-\beta)\int_0^t e^{-\alpha u}I_o(\beta u)\,du\right]H(t) \quad .\end{aligned} \tag{A.2}$$

In the foregoing expression, $I_o$ is the modified Bessel function of zero order, and $H(t)$ is the step function. If we make the identification

$$\begin{aligned}\alpha &= \frac{\alpha_1+\alpha_2}{2} = \frac{1}{2\mu}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)k^2 \quad , \\ \beta &= \frac{\alpha_1-\alpha_2}{2} = \frac{1}{2\mu}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)k^2 \quad ,\end{aligned} \tag{A.3}$$

we can write (suppressing the $k$ dependence)

$$\bar{Q}(s) = (s+\alpha_1)\bar{R}(s) \quad . \tag{A.4}$$

We then have

$$\begin{aligned}\underline{L}^{-1}\bar{Q}(s) &= \underline{L}^{-1}\left((s+\alpha_1)\bar{R}(s)\right) = \hat{Q}(t) \quad , \\ \hat{Q}(t) &= \alpha_1\hat{R}(t) + \frac{d}{dt}\hat{R}(t) \quad .\end{aligned} \tag{A.5}$$

Substituting equation (A.2) into equation (A.5) yields

$$\begin{aligned}\hat{Q}(t) &= e^{-\alpha t}\left[\beta I_1(\beta t) + \alpha I_o(\beta t)\right] \\ &\quad + (\alpha^2-\beta^2)\int_0^t e^{-\alpha u}I_o(\beta u)\,du + \delta(t) \quad .\end{aligned} \tag{A.6}$$

## APPENDIX B: APPLICATION OF FALTUNG THEOREM

In the main body of the report we are concerned with calculations of the form

$$f_{12}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x-y) f_2(y)\, dy \tag{B.1}$$

where $f_2(y)$ is given by

$$f_2(y) = \frac{\partial^2}{\partial y^2} g(y) \quad . \tag{B.2}$$

For computational purposes it may be desirable to organize the computation of $f_{12}(x)$ so that only the function $g(y)$ appears in the integration, and none of its derivatives. This is accomplished in the following way: We let

$$g'(y) = \frac{\partial g}{\partial y} \tag{B.3}$$

so that we can write

$$f_2(y) = \frac{\partial}{\partial y}\, g'(y) \quad . \tag{B.4}$$

Substituting equation (B.4) in equation (B.1) gives

$$f_{12} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x-y)\, \frac{\partial g'}{\partial y}\, dy \quad . \tag{B.5}$$

We now integrate equation (B.5) by parts to give

$$\begin{aligned} f_{12} &= \frac{1}{2\pi} \left[ g'(y) f_1(x-y) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(y)\, \frac{\partial f_1(x-y)}{\partial y}\, dy \right] \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} g'(y)\, \frac{\partial f_1(x-y)}{\partial y}\, dy \quad . \end{aligned} \tag{B.6}$$

Letting

$$z = x - y \quad , \tag{B.7}$$

we have

$$\frac{\partial}{\partial y} = \frac{\partial f_1}{\partial z}\,\frac{\partial z}{\partial y} = -\frac{\partial f_1}{\partial z} \quad . \tag{B.8}$$

There results

$$f_{12}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g'(y)\left(\frac{\partial f_1(z)}{\partial z}\right)_{z=x-y} dy \quad . \tag{B.9}$$

By executing the same procedure once more we obtain the desired result

$$f_{12}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\left(\frac{\partial^2 f_1(z)}{\partial^2 z}\right)_{z=x-y} dy \quad . \tag{B.10}$$

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